FIGURES OF MERIT FOR MULTI-STAGE CRYOCOOLERS

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ABSTRACT

The "coefficient of performance" (CoP) is often used as a measure of efficiency for single-stage cryocoolers, but such a parameter is not well defined for multi-stage cryocoolers. We propose a simple definition of an electrical "figure of merit" (FoM) representative of the distributed refrigeration power of multi-stage cryocoolers, that resolves this issue for applications where heat-sinking of power and signal leads at intermediate stages is an important end-user requirement. Two cases are considered which yield somewhat different results. A Power Lead FoM (PL-FoM) is derived, based on the largest electric current that can be flowed from ambient to the lowest temperature stage. A Signal Lead FoM (SL-FoM) is also derived, based on achieving minimum electrical attenuation on the signal leads. Each FoM represents a temperature-weighted combination of the heat lifts of the various stages. The two FoMs can aid in the selection of an optimal multi-stage cryocooler for operation of superconducting devices, for example.

KEYWORDS: CoP, efficiency, current leads, signal attenuation

INTRODUCTION

There are several standard ways to evaluate and compare cryocoolers. One common practical figure is the available heat lift at the preferred operating temperature $Q_{lift}(T_{op})$. Further, the relative efficiency of such a cryocooler is often expressed as the "Coefficient of Performance" (CoP), the ratio of the heat lift at T_{op} to the input electrical power at room temperature:

$$CoP = \mathrm{Q}_{lift}(T_{op}) \big/ P_{RT} \tag{1}$$

However, a multi-stage cryocooler has more internal degrees of freedom, and the values of heat lift at intermediate stages are also important. This is particularly true when a cryocooler supports the operation of superconducting devices, where the thermal balance is dominated not by heat dissipation of the superconducting device itself, but rather by heat conduction on electrical leads [1]. If leads coming from room temperature are thermally anchored to each progressively lower temperature stage, then the heat load to the coldest stage will be minimized. Since heat extraction always becomes more difficult at lower temperatures, this approach makes the most efficient use of cooling resources. It would be desirable to define simple figures of merit for the *whole* multi-stage cryocooler in a way that permits one to compare the expected performance of different cooler designs.

As we can distinguish cryocooled systems with electrical leads dominated by either power leads or signal transmission lines, we shall derive two different types of FoM. A Power Lead FoM (PL-FoM) that gives the maximum current that can be carried down to the operating temperature and a Signal Lead FoM (SL-FoM) that represents the minimum signal attenuation between room temperature and operating temperature.

POWER LEAD FOM

We start with the observation that there is a minimum quantity of heat conducted on optimized bias leads between stages at T_{hot} and T_{cold} which depends only on the endpoint temperatures and on the bias current I_b, regardless of the number and composition of the leads [1].

$$Q_{min} = (2\pi/\surd 3)\,(I_b k_B/e)\,(T_{hot}^2 - T_{cold}^2)^{0.5} \approx 3.6\, I_b k_B T_{hot}/e \qquad (2)$$

where e is the electron charge and k_B the Boltzmann constant; the latter approximation is valid in the usual case that $T_{cold} << T_{hot}$. This relation is quite general, for any normal metallic leads with proportionality between electrical and thermal resistance (obeying the Wiedemann-Franz law). This law is valid for simple resistive metals at high temperature and when the resistance is impurity limited at low temperature. The greatest deviations from the Wiedemann-Franz law are for ultra-pure metals at low temperature, not generally relevant for practical materials for electrical leads. In most practical cases, deviations are likely to be relatively small, so that the FoMs derived using this approximation should also be reasonably valid.

One can now turn equation (2) around and define an equivalent current rating I_n associated with the n^{th} stage:

$$I_n = Q_n e/[3.6 k_B (T_{n-1}^2 - T_n^2)^{0.5}] \approx Q_n e/[3.6 k_B T_{n-1}] \qquad (3)$$

where Q_n is the available heat lift for the n^{th} stage and T_{n-1} is the temperature of the next warmer stage. This represents the largest bias current that the stage could support, if this were the only source of heat. Note that to maintain the same value of I_n, the heat lift Q_n must increase proportionally to T_{n-1}. This indicates the importance of intermediate stages.

The overall performance of the multi-stage cryocooler reflects the performance of its weakest link. Hence one can define the overall current rating for the cryocooler as:

$$I_{eff} = \min(I_n) \qquad (4)$$

Note that depending on the specific type of cooler, or due to the specificity of the application, the weakest stage is not necessarily the coldest. This also suggests that in optimizing the design of the various stages, there may be little advantage in having substantial excess capacity (i.e., $I_n >> I_{eff}$ for any n). On the other hand, excess cooling capacity on the first (warmest) stage might be used for other purposes, such as shielding of room-temperature thermal radiation or mounting of low-noise semiconductor amplifiers.

This current rating I_{eff} is relevant not only for cryocoolers for microelectronic applications, but also for high-current devices such as superconducting magnets. In general, the bias current may not exceed I_{eff} for normal-metallic leads. However, the bias current may exceed I_n for colder stages if the input line is a properly designed superconducting lead [2, 3], which does not obey the Wiedemann-Franz law. Finally, one can define the relative efficiency of a given cryocooler by the relation:

$$F_{PL} = {}^{I_{eff}}\!\big/_{P_{RT}} = \min\left\{ {}^{Q_n e}\!\big/_{(3.6\,\mathrm{k_B T_{n\text{-}1}})P_{RT}} \right\} \qquad (5)$$

It is important to note that both I_{eff} and F_{PL} are defined for a specific temperature and cooling distribution on the stages of the cooler. This FoM has to be evaluated again if one considers a different working temperature on any stage. This PL-FoM has units of A/W, unusual for efficiency, but this should be useful in comparing different cryocooler designs, including those with different numbers of stages. This is representative of the ability of a cryocooled system to carry a certain amount of current to the coldest stage.

However, if we consider the FoM as a strict measure of the efficiency, it may be convenient to define an effective cooling power Q_{PL} that scales in the same way as I_{eff}. Then by stripping off all constants from Eq. (5) and normalizing to ambient temperature T_0 we have:

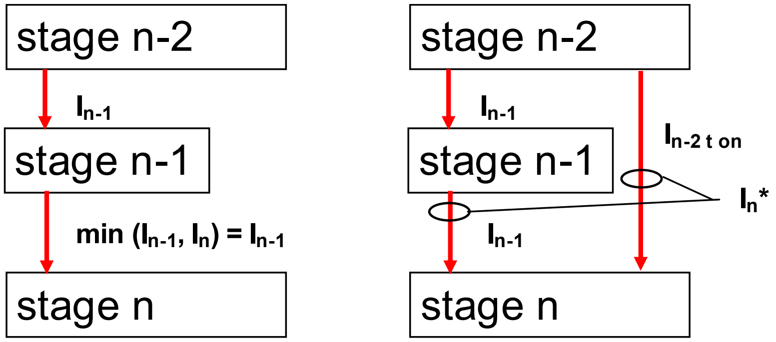
$$Q_{PL} = \min(T_0 Q_n / T_{n-1}) \qquad (6)$$

This permits one to compare the effective cooling capacities of alternative cryocooler types and designs. One can also extend this to an effective dimensionless efficiency by dividing by the total compressor power at room temperature:

$$\eta_{PL} = Q_{PL}/P_{RT} = (T_0/P_{RT}) \min(Q_n/T_{n-1}) \qquad (7)$$

Note that just like the CoP, it would not be appropriate to use these parameters to compare a 4 K cryocooler with a 70 K cryocooler, regardless of the number of stages, or to compare any coolers that don't have the same operating temperature.

For most multi-stage coolers, the cooling power is lower at lower temperature stages; however, in some specific cases, the cooling power available on an intermediate stage may be similar to or lower than that of the final stage. This is, for example, the case when heat intercepts [4] are mounted on the cooler or when the stage is heavily loaded by another application such as low temperature filters or amplifiers. In this specific case, because of the weakness of the n-1th stage, the current I_{n-1} one can flow between stages n-1 and n-2 is lower than the current I_n that can be flowed between stages n-1 and n. The FoM is thus limited by the intermediate stage and not by the coldest stage. This is clearly not ideal for thermalization of power leads, as reflected by the low value of the FoM. If this design cannot be changed, then in order to improve the efficiency of the system (and so the FoM), one should also consider the possibility of flowing some current directly from the n-2th stage to the nth stage (see FIGURE 1). On the left-hand drawing, we can see that the current that can be flowed down to operating temperature is limited by I_{n-1}. On the right side, in order to increase the total current and thus the PL-FoM, some current is flowed directly from the n-2th stage to the nth stage.



$I_n >> I_{n-1}$ and $I_{n-2} >> I_{n-1}$

Current flow without bypassing Current flow with bypassing

FIGURE 1: Parallel thermalization to optimize the PL-FoM

For the PL-FoM calculation, this is reflected by using I_n* as the current, equal to the sum of the current flowing from n-1 to n and that flowing from n-2 to n: $I_n{*}=I_{n-1}+I_{(n-2)\text{-to-}n}$

So, by calculating I_{n-1} and $I_{(n-2)\text{-to-}n}$ using relation (2) we obtain:

$$I_n{*} = {}^{e}\!\big/_{3.6k_B}\left[\frac{Q_n}{T_{n-1}}+\frac{Q_n}{T_{n-2}}\left(\frac{I_n - I_{n-1}}{I_n}\right)\right] \tag{8}$$

We thus have:

$$I_n{*} = {}^{e}\!\big/_{3.6k_B}\left[\frac{Q_n}{T_{n-1}}+\frac{Q_n}{T_{n-2}}\left(1-\frac{Q_{n-1}.T_{n-1}}{Q_n.T_{n-2}}\right)\right] \tag{9}$$

As a more general calculation of the current flowing to the n^{th} stage, I_n* is now considered instead of I_n and I_{n-1} in (4) for the determination of a revised FoM.

SIGNAL LEAD FoM

Now let us consider another limit that may be relevant in some cases. For example, a cryogenically cooled sensor array for imaging, where there may be a large number of output lines extending from cryogenic temperatures, each carrying a weak signal. The significance of weak signals is that signal attenuation up to room temperature must be minimized, and also that Joule heating from the signal can be neglected. If the cooler is to be used for cooling such systems, in which thermal conduction on the transmission lines is the dominant thermal load, one can calculate a Signal Lead Figure of Merit (SL-FoM) for this cryocooler.

This calculation is based on the following principle. We calculate the maximum electrical conductance permitted for (electrically) connecting ambient temperature to the coldest stage and each stage in between, subject to the distributed refrigeration power available. We consider that the Wiedemann-Franz relation is obeyed, and so we can express the thermal conductivity as proportional to the electrical conductivity.

This maximum electrical conductance corresponds to the minimum attenuation we can have, since attenuation is proportional to resistance. For this analysis, we consider that the power dissipated by the Joule effect is small enough to be neglected.

Consider for simplicity a conductor with a constant cross section S and a total length *l*, split into lengths l_i between adjacent stages. We then have: $l = \Sigma l_i$ and thus $R = \Sigma R_i$. The heat load on the ith stage is defined by:

$$Q_i = \frac{S}{l_i} \int_{Ti-1}^{Ti} \kappa dT \qquad (10)$$

where κ is the thermal conductivity.

The electrical resistance between two stages can be written:

$$R_i = \frac{l_i}{S} \int_{Ti-1}^{Ti} \rho dT \qquad (11)$$

where ρ is the electrical resistivity.

By combining these two equations and using the Wiedemann-Franz Law we obtain:

$$R_i = \frac{L\,(T_{i-1}{}^2 - T_i{}^2)}{2Q_i} \qquad (12)$$

where $L = \pi^2 k_B{}^2/3e^2$ is the Lorenz constant.

For a given power Q_i on the ith stage of the cooler, we have determined the minimum resistance of the link between the ith and i-1th stages, provided it is implemented by a material obeying the Wiedemann-Franz Law. By summing the different R_i calculated for this cooler, we have the minimum resistance from room temperature to operating temperature that this cooler can sustain.

The total attenuation of a high-frequency signal on a transmission line with characteristic impedance Z_0 is proportional to the series resistance on the line [7], and is given by:

$$\alpha_i \approx R_i/2Z_0 \qquad (13)$$

In order to have a Figure of Merit inversely proportional to the attenuation, we consider the inverse of the resistance. The SL-FoM of a cryocoolers can then be written:

$$F_{SL} = \left[\sum_i \frac{L\,(T_{i-1}{}^2 - T_i{}^2)}{2Q_i} \right]^{-1} \cdot \frac{1}{P_{RT}} \qquad (14)$$

This is expressed in Siemens per Watt ($S.W^{-1}$), again a unit unusual for efficiency, but that gives a good relative estimate of the capability of the system to conduct a weak high-frequency signal between the cryogenic operating temperature and room temperature.

Similarly to the Power Lead FoM analysis, if we consider the FoM as a strict measure of the efficiency, the FoM can be made dimensionless.

If we consider:

$$(T_{i-1}{}^2 - T_i{}^2)^{0.5} \approx T_{i-1} \text{ if } T_i << T_{i-1} \qquad (15)$$

From Eq. (13), one may define an effective cooling power for the multi-stage cooler by:

$$Q_{SL} = T_0^2/\Sigma[T_{i-1}^2/Q_i] \approx 1/\Sigma[(T_{i-1}/T_0)^2/Q_i], \qquad (16)$$

where the latter approximation is for the common case that $T_i << T_{i-1}$ for all the stages. Similarly, one may define an efficiency parameter:

$$\eta_{SL} = Q_{SL}\Big/P_{RT} = \left[\sum_i \frac{(T_{i-1}{}^2 - T_i{}^2)}{Q_i}\right]^{-1} \cdot \frac{T_0{}^2}{P_{RT}} \qquad (17)$$

The SL-FoM can easily be used for determining the number of signal lines one can mount onto a cooler. One can define a line by its attenuation; the resistance of the line can then be written according to (13): $R \approx \alpha.2Z_0$

If we define β as the ratio between the effective cross-sectional areas of the shielding and the central conductor of one coaxial line, we can calculate the maximum number of lines for the cooler as:

$$n = \sum_i \frac{L\,(T_{i-1}{}^2 - T_i{}^2)}{2Q_i} \frac{1}{\beta\alpha 2Z_0} \qquad (18)$$

Similarly to the Power Lead Figure of Merit, this analysis is optimized if the limiting stage is the coldest one. In other words, if the cooler has an intermediate weak stage, an optimization of the operating temperature of this stage may be necessary to have a good Figure of Merit.

It is important to note that the formula of the SL-FoM in Eq. (14) does not take into account the skin effect that takes place in conductors for high-frequency signals. The full inclusion of the skin effect into the SL-FoM analysis leads to complex equations beyond the scope of the present paper. However, for conductors with constant metallic thickness d, we can identify two key limits, depending on the size of the skin depth δ, given by:

$$\delta = \sqrt{(2\rho/\omega\mu)} \qquad (19)$$

where ρ is electric resistivity, ω is signal angular frequency, μ is magnetic permeability.

The skin effect essentially constrains high frequency currents to flow within δ of the surface. In the low-frequency limit where d <<δ, one can essentially ignore the skin effect, and the SL-FoM follows as before. In the high-frequency limit where d >> δ, all electrical resistances (including the series resistance of the signal leads between each stage of the cryocooler) are increased by a factor γ ~ d/δ. This will tend to increase all attenuations by a same factor γ. If γ is the same over the temperature range of the cooler, then the overall SL-FoM will continue to apply. On the other hand, if γ is a strong function of temperature, then the analysis must be substantially revised. From Eq. (19), we can see that δ and thus γ are constant if the resistivity is independent of temperature. For an alloy such as stainless steel then, the SL-FoM should remain generally valid (although with an increased attenuation factor). This may be the case in practical systems with many signal leads, since the alloy conductors also minimize the inter-stage heat leak. For pure copper in the high-frequency limit, on the other hand, the SL-FoM would need to be revised.

As a result we can say that the optimization of the resistance just like the determination of an SL-FoM is valid for systems with negligible skin effect, as well as for any system where the electrical resistivity of the conductor is constant over the temperature range of the cooler.

EXAMPLE OF APPLICATION

The different figures of merit described above can easily be derived for any cryocooler, using the specifications provided by the manufacturer. Further, if some power dissipation at an intermediate stage is needed or foreseen (due to electrical heating, heat exchanger, etc.…), one just has to subtract this amount of power from the available power before calculating the FoM of the cryocooled system. FIGURE 2 shows an example of some parameters to be considered when calculating the FoM for a specific system.

The discrete heat load calculated on each stage is then removed from the available cooling power of the stage. For example for the first stage one thus has:

$$Q_{1\text{st stage}} = Q_{\text{spec}} - (Q_{\text{filters}} + Q_{\text{rad}} + \ldots) \tag{20}$$

The FoM can then be calculated, and be used for the selection of the cooler for the application.

The tables below show the PL-FoM and SL-FoM of different pulse-tube cryocoolers from Cryomech, with no additional heat load on any stages. The data are taken from the Cryomech website [5] and from work presented in 2007 [6]. TABLE 1 shows the different parameters used for the calculation of the FoM and the different FoM calculated for each cooler. TABLE 2 shows the intermediate values found when calculating the FoM.

Looking at TABLE 1, generally speaking, for the same number of stages, the more powerful the cooler, the more efficient it is. This is also reflected in the CoP. However, the 3-stage cooler has the worst CoP, but reasonably good FoM. The benefit of a third stage, which is intuitive but cannot be quantified by the CoP, is clearly visible in the FoM.

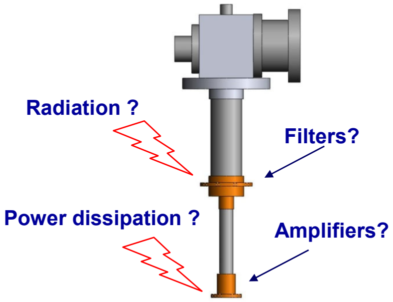


FIGURE 2: Calculating the FoM of a complete system

TABLE 1: FoM for Cryomech Cryocoolers (pulse tubes)

Model	Input power	T_1	Q_1	T_{int}	Q_{int}	T_{cold}	Q_{cold}	SL-FoM	PL-FoM	CoP
PT410	7.2 kW	45 K	35 W	NA	NA	4.2 K	1 W	2.3 S/W	10 A/W	$1.4\text{x}10^{-4}$
PT405	5.4 kW	65 K	25 W	NA	NA	4.2 K	0.5 W	0.4 S/W	4.6 A/W	$9.3\text{x}10^{-5}$
PT3S403	10 kW	44 K	37 W	19 K	19 W	4.2 K	0.3 W	2.3 S/W	5.2 A/W	$3\text{x}10^{-5}$

TABLE 2: FoM calculation details

Model	PL-FoM				SL-FoM			
	I_1	I_{int}	I_{cold}	I_{eff}	R_1	R_{int}	R_{cold}	R_{tot}
PT410	380 A	NA	71.9 A	71.9 A	$3.1 \times 10^{-5}\,\Omega$	NA	$2.5 \times 10^{-5}\,\Omega$	$5.5 \times 10^{-5}\,\Omega$
PT405	274.9 A	NA	24.8 A	24.8 A	$4.2 \times 10^{-5}\,\Omega$	NA	$1.0 \times 10^{-4}\,\Omega$	$1.5 \times 10^{-4}\,\Omega$
PT3S403	401.5 A	1542 A	52.1 A	52.1 A	$2.9 \times 10^{-5}\,\Omega$	$1 \times 10^{-6}\,\Omega$	$1.4 \times 10^{-5}\,\Omega$	$4.4 \times 10^{-5}\,\Omega$

In TABLE 2, we can clearly see that the PL-FoM is limited by the coldest stage of the cooler. This is due to the excess cooling power available on the other stages. The SL-FoM however receives contributions of about the same amount from all sections of the cooler. This difference is explained by the characteristic of each FoM. For the SL-FoM, the cooling power of each stage is fully used, as the SL-FoM is not linked to one limiting factor. For the PL-FoM on the other hand only the lowest current is considered, thus leading to extra current capacity (extra cooling power) on the other stages. This also suggests that there is an optimum for the cooling power of each stage, corresponding to a balance in the distribution of the cooling power of the whole cooler. From the form of the equations of the PL-FoM (Eq. 5) and the SL-FoM (Eq. 14), we can say that to optimize a cooler in regard to these FoMs, the cooling power on the i^{th} stage should scale as T_{i-1} for the PL-FoM and T_{i-1}^2 for the SL-FoM.

CONCLUSION

We have proposed two figures of merit (for power leads and signal leads) to provide a fair comparison of the thermal performance of cryocoolers, even if they don't have the same number of stages and don't have the same intermediate temperatures. These ratings can be derived easily for any cooler, using the manufacturer's datasheet, and can be used to optimize one's specific application. They are useful for a cryocooler designer who wants to maximize the global efficiency of the system or for an end-user as a guide in the choice of the cryocooler and the design of the cryopackage. Further work will be done in order to show how one can optimize the operating point of a cooler in order to get the best FoM possible.

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